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Behaviour of Exponential Splines as Tensions Increase without Bound

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Schweikert (*J. Math. Phys.* **45** (1966), 312–317) showed that for sufficiently high tensions an exponential spline would have no more changes in sign of its second derivative than there were changes in the sign of successive second differences of its knot sequence. Späth (*Computing* **4** (1969), 225–233) proved the analogous result for first derivatives, assuming uniform tension throughout the spline. Later, Pruess (*J. Approx. Theory* **17** (1976), 86–96) extended Späth's result to the case where the inter-knot tensions p_i may not all be the same but tend to infinity at the same asymptotic growth rate, in the sense that $p_i \in \Theta(p_1)$ for all i. This paper extends Pruess's result by showing his hypothesis of uniform boundedness of the tensions to be unnecessary. A corollary is the fact that for high enough minimum interknot tension, the exponential spline through monotone knots will be a $\mathscr{C}^2$ monotone curve. In addition, qualitative bounds on the difference in slopes between the interpolating polygon and the exponential spline are developed, which show that Gibbs-like behaviour of the spline's derivative cannot occur in the neighbourhood of the knots. © 1997 Academic Press

1. NOTATION AND BASIC FACTS ABOUT NATURAL EXPONENTIAL SPLINES

It will be convenient to use the notation $\text{Int}[a_1, ..., a_m]$ for the smallest closed interval containing $a_1, ..., a_m$ and $\text{Int}(a_1, ..., a_m)$ for its interior. Let $\Delta = ((x_i, y_i))_{i=0, 1, ..., n}$ be a sequence of plane points with $x_0 < x_1 < \cdots < x_n$, hereafter called the *knots*.

In what follows, the ith subinterval means $[x_{i-1}, x_i]$ and the ith knot means (x_i, y_i). All quantities related to the spline in tension in this subinterval or at this knot will bear the subscript i. Thus for example h_i will denote $x_i - x_{i-1}$ and m_i will denote $(y_i - y_{i-1})/(x_i - x_{i-1})$, the slope of the

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interpolating polygon over the ith subinterval; at interior knots, d_i will denote the second divided difference: $(m_{i+1}-m_i)/(h_i+h_{i+1})$; since this paper deals only with natural splines it will serve our purposes best to define $d_0=0=d_n$.

For each $i=1, ..., n$ let p_i: $[0, \infty)\to[0, \infty)$ be any given non-negative function with the following properties:

$$p_i(\xi)=0 \qquad \text{if and only if} \quad \xi=0, \tag{1.1a}$$

$$p_i \text{ is continuous,} \tag{1.1b}$$

$$\lim_{\xi\to\infty} p_i(\xi)=\infty. \tag{1.1c}$$

It has been shown [8] that the exponential spline, as defined in the next paragraph, in many cases is an adequate approximation to the true mechanical spline with tension supplied at the knots (sometimes referred to in the literature [1] as "interpolation by elastica") through the same data points with tension $p_i^2(\xi)$ on the ith subinterval. Thus, although it is not dimensionally correct, we will refer to $\mathbf{p}(\xi):=(p_1(\xi), ..., p_n(\xi))^t$ as the *tension strategy* (for smoothly increasing the inter-knot tensions from 0 to ∞).

One set of defining properties for the natural exponential spline $\tau_\xi(x)$, through the knot sequence Δ, with tension strategy $\mathbf{p}$, is:

(i) $\tau''_\xi(x_0)=\tau''_\xi(x_n)=0$,

(ii) τ_ξ interpolates the (x_i, y_i) for $i=0, 1, ..., n$,

(iii) τ'_ξ is continuous across x_i for $i=1, 2, ..., n-1$,

(iv) τ''_ξ is continuous across x_i for $i=1, 2, ..., n-1$,

(v) among the functions in[1] $\mathscr{K}^2[x_0, x_n]$ satisfying (i)–(iv) above, τ_ξ minimizes

$$\sum_{i=1}^{n}\int_{x_{i-1}}^{x_i} [\tau''_\xi(x)]^2+p_i^2[\tau'_\xi(x)-m_i]^2\,dx.$$

The idea behind this definition is that we wish not only to minimize curvature (approximated by τ''_ξ) but also to keep the slope of the curve close to that of the interpolating polygon. So we minimize a weighted average of $\int(\tau''_\xi)^2$ and $\int(\tau'_\xi-m_i)^2$ where the weight p_i^2 selected determines how flat the curve $\tau_\xi(x)$ will be between the knots, at the expense of kinkiness at the knots.

[1] $\mathscr{K}^2[a, b]$ are those functions whose first derivative is absolutely continuous, and whose second derivative is L^2 on $[a, b]$.

We can derive from (v), by the calculus of variations that on $[x_{i-1}, x_i]$

$$\tau_\xi^{(4)}(x) = p_i^2(\xi)\, \tau_\xi''(x), \qquad i = 1, 2, ..., n. \tag{1.2}$$

Conditions (i)–(iv) become the $4n$ boundary conditions:

$$\tau_\xi(x_0) = y_0 \tag{1.3a}$$

$$\tau_\xi''(x_0) = 0 \tag{1.3b}$$

$$\tau_\xi(x_i^-) = y_i \tag{1.3c}$$

$$\tau_\xi(x_i^+) = y_i \tag{1.3d}$$

$$\tau_\xi'(x_i^-) = \tau_\xi'(x_i^+) \tag{1.3e}$$

$$\tau_\xi''(x_i^-) = \tau_\xi''(x_i^+) \tag{1.3f}$$

$$\tau_\xi(x_n) = y_n \tag{1.3g}$$

$$\tau_\xi''(x_n) = 0 \tag{1.3h}$$

where $i = 1, ..., n-1$.

By the *moments* of the spline we shall mean the second derivatives $M_i(\xi) = \tau_\xi''(x_i)$, $i = 0, ..., n$, at the knots. These are well-defined because of boundary condition (1.3f).

Define $\sigma_i(\xi) = h_i\, p_i(\xi)$ for $i = 1, ..., n$.

For $i = 1, ..., n$ define the *slackness* of τ_ξ on $[x_{i-1}, x_i]$ by:

$$\rho_i(\xi) = \begin{cases} \dfrac{\tanh \sigma_i(\xi)}{\sigma_i(\xi)} & \text{if} \quad \sigma_i(\xi) > 0 \\ 1 & \text{if} \quad \sigma_i(\xi) = 0. \end{cases} \tag{1.4}$$

Thus ρ decreases from 1 (inclusive) to 0 (exclusive) as σ increases from 0 (inclusive) to ∞ (exclusive). In this paper p_i, p_i^2, σ_i, and ρ_i will be used interchangeably as measures of tension on $[x_{i-1}, x_i]$. An advantage of viewing τ as being determined by σ and Δ (or by ρ and Δ) rather than by $\mathbf{p}$ and Δ is that then, for σ fixed, τ is invariant under any affine transformation $\mathscr{T}$ of $\mathbb{R}^2$, in the sense that $\mathscr{T}((x, \tau(x))$ is the graph of the exponential spline through knots $\mathscr{T}(\Delta)$. An advantage of using ρ rather than σ for tension is that the cubic interpolating spline then becomes a special member of the family of exponential splines (when $\rho = 1$) rather than as a limit of the family (as $\sigma \to 0$).

For each value of $\rho \in (0, 1]$ it will be convenient to have available notation for the following four functions. Define $f_\rho\colon (0, 1] \to [0, \infty)$, $F_\rho\colon [0, \infty) \to (0, 1]$, $Q_\rho\colon [0, 1] \to \mathbb{R}$, and $P_\rho\colon [0, 1] \to \mathbb{R}$ by:

$$f_\rho(\theta)=\begin{cases}\dfrac{\sinh(\sigma(1-\theta))}{\sinh(\sigma\theta)} & \text{if } 0<\rho<1,\\[2mm] \dfrac{1-\theta}{\theta} & \text{if } \rho=1;\end{cases} \tag{1.5}$$

$$F_\rho(y)=\begin{cases}\dfrac{1}{2\sigma}\ln\left(\dfrac{y+e^{\sigma}}{y+e^{-\sigma}}\right) & \text{if } 0<\rho<1,\\[2mm] \dfrac{1}{1+y} & \text{if } \rho=1;\end{cases} \tag{1.6}$$

$$Q_\rho(\theta)=\begin{cases}\left(\dfrac{\cosh(\sigma\theta)}{\sinh\sigma}-\dfrac{1}{\sigma}\right)\coth\sigma & \text{if } 0<\rho<1,\\[2mm] \dfrac{3\theta^2-1}{6} & \text{if } \rho=1;\end{cases} \tag{1.7}$$

$$P_\rho(\theta)=\begin{cases}\left(\dfrac{\sinh(\sigma\theta)}{\sinh\sigma}-\theta\right)\dfrac{\coth\sigma}{\sigma} & \text{if } 0<\rho<1,\\[2mm] \dfrac{\theta^3-\theta}{6} & \text{if } \rho=1.\end{cases}$$

We remark that f_ρ and F_ρ are strictly decreasing bijections. In fact, they are inverses. Also, $Q_\rho(\theta)=P'_\rho(\theta)$ and $f_\rho(\theta)=Q'_\rho(1-\theta)/Q'_\rho(\theta)$. For a graph of $Q_\rho(\theta)$ see Fig. 2.

Returning now to the derivation of $\tau_\xi(x)$, we get by solving (1.2) that for all $x\in[x_{i-1},x_i]$

$$\tau_\xi(x)=A_i\sinh\left(\sigma_i\frac{x-x_{i-1}}{h_i}\right)+B_i\left[\cosh\left(\sigma_i\frac{x-x_{i-1}}{h_i}\right)-1\right]$$
$$+C_i\frac{x-x_{i-1}}{h_i}+D_i \qquad \text{for} \quad i=1,...,n \tag{1.8}$$

subject to the boundary conditions (1.3a)–(1.3h), which become, on solving for A_i, B_i, C_i, D_i:

$$A_i=\frac{M_i-M_{i-1}\cosh\sigma_i}{(\sigma_i^2/h_i^2)\sinh\sigma_i}, \tag{1.9a}$$

$$B_i=M_{i-1}\frac{h_i^2}{\sigma_{i1}^2}, \tag{1.9b}$$

$$C_i=y_i-y_{i-1}-\frac{h_i^2}{\sigma_i^2}(M_i-M_{i-1}), \tag{1.9c}$$

$$D_i=y_{i-1} \tag{1.9d}$$

where $i=1,...,n$.

Here $M_0 = 0 = M_n$ while $M_1, ..., M_{n-1}$ are the solutions of the system

$$\begin{pmatrix} \delta_1 & \nu_1 & 0 & \cdots & \cdots & 0 \\ \mu_2 & \delta_2 & \nu_2 & \ddots & & \vdots \\ 0 & \mu_3 & \delta_3 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \delta_{n-2} & \nu_{n-2} \\ 0 & \cdots & \cdots & 0 & \mu_{n-1} & \delta_{n-1} \end{pmatrix} \begin{pmatrix} M_1 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ M_{n-1} \end{pmatrix} = \begin{pmatrix} d_1 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ d_{n-1} \end{pmatrix} \tag{1.10}$$

and where:

$$\mu_i = -\frac{h_i}{h_i + h_{i+1}} \rho_i Q_{\rho_i}(0) \qquad \text{for} \quad i = 2, ..., n-1,$$

$$\delta_i = \frac{h_i}{h_i + h_{i+1}} \rho_i Q_{\rho_i}(1) + \frac{h_{i+1}}{h_i + h_{i+1}} \rho_{i+1} Q_{\rho_{i+1}}(1) \qquad \text{for} \quad i = 1, ..., n-1,$$

$$\nu_i = -\frac{h_{i+1}}{h_i + h_{i+1}} \rho_{i+1} Q_{\rho_{i+1}}(0) \qquad \text{for} \quad i = 1, ..., n-2.$$

So (1.8)–(1.10) provide a system of formulas for computing $\tau_\xi(x)$, at least theoretically.

The following notation and formulas will be of use later. For $i = 1, ..., n-1$ define

$$L_i(\xi) = M_i(\xi) \left(\frac{h_i}{h_i + h_{i+1}} \rho_i(\xi) + \frac{h_{i+1}}{h_i + h_{i+1}} \rho_{i+1}(\xi) \right). \tag{1.11}$$

The system (1.10) for the moments $M_i(\xi)$ with $i = 1, ..., n-1$ can then be rewritten as the linear system

$$\mathbf{R}_\xi \mathbf{L}(\xi) = \mathbf{d} \tag{1.12}$$

in the new variables $L_i(\xi)$, where

$$\mathbf{L}(\xi) = \begin{bmatrix} L_1(\xi) \\ \vdots \\ L_{n-1}(\xi) \end{bmatrix}, \qquad \mathbf{d} = \begin{bmatrix} d_1 \\ \vdots \\ d_{n-1} \end{bmatrix},$$

$$\mathbf{R}_\xi = \begin{pmatrix} \beta_1 & \gamma_1 & 0 & \cdots & \cdots & 0 \\ \alpha_2 & \beta_2 & \gamma_2 & \ddots & & \vdots \\ 0 & \alpha_3 & \beta_3 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \beta_{n-2} & \gamma_{n-2} \\ 0 & \cdots & \cdots & 0 & \alpha_{n-1} & \beta_{n-1} \end{pmatrix}$$

with

$$\alpha_i = -\frac{h_i\ \rho_i}{h_{i-1}\ \rho_{i-1}+h_i\ \rho_i}\cdot\frac{h_{i-1}+h_i}{h_i+h_{i+1}}\cdot Q_{\rho_i}(0),$$

$$i=2, ..., n-1,$$

$$\beta_i = \frac{h_i\ \rho_i}{h_i\ \rho_i+h_{i+1}\ \rho_{i+1}}\cdot Q_{\rho_i}(1)+\frac{h_{i+1}\ \rho_{i+1}}{h_i\ \rho_i+h_{i+1}\ \rho_{i+1}}\cdot Q_{\rho_{i+1}}(1),$$

$$i=1, ...,n-1,$$

$$\gamma_i = -\frac{h_{i+1}\ \rho_{i+1}}{h_{i+1}\ \rho_{i+1}+h_{i+2}\ \rho_{i+2}}\cdot\frac{h_{i+1}+h_{i+2}}{h_i+h_{i+1}}\cdot Q_{\rho_{i+1}}(0),$$

$$i=1, ..., n-2.$$

Let us agree to use $\theta_i \in [0,1]$ to represent the relative position of x within the interval $[x_{i-1}, x_i]$, that is, $\theta_i = (x - x_{i-1})/h_i$. Then, substituting into the derivative of (1.8) the expressions (1.9a)–(1.9d) and (1.11) yields, for $x \in [x_{i-1}, x_i]$ and $i = 1, ..., n$,

$$\tau'_\xi(x) = m_i - L_{i-1}(\xi)\cdot(h_{i-1}+h_i)\cdot\frac{h_i\rho_i(\xi)}{h_{i-1}\ \rho_{i-1}(\xi)+h_i\ \rho_i(\xi)}\cdot Q_{\rho_i}(1-\theta_i)$$
$$+L_i(\xi)\cdot(h_i+h_{i+1})\cdot\frac{h_i\ \rho_i(\xi)}{h_i\ \rho_i(\xi)+h_{i+1}\ \rho_{i+1}(\xi)}\cdot Q_{\rho_i}(\theta_i) \qquad (1.13)$$

and hence

$$\tau''_\xi(x) = L_{i-1}(\xi)\cdot(h_{i-1}+h_i)\cdot\frac{1}{h_{i-1}\ \rho_{i-1}(\xi)+h_i\ \rho_i(\xi)}\cdot\frac{\sinh(\sigma_i(\xi)(1-\theta_i))}{\sinh\sigma_i(\xi)}$$
$$+L_i(\xi)\cdot(h_i+h_{i+1})\cdot\frac{1}{h_i\ \rho_i(\xi)+h_{i+1}\ \rho_{i+1}(\xi)}\cdot\frac{\sinh(\sigma_i(\xi)\theta_i)}{\sinh\sigma_i(\xi)} \qquad (1.14)$$

and (see Fig. 1)

$$\tau_\xi(x) = (1-\theta_i)\ y_{i-1}+\theta_i\ y_i$$
$$+h_i\cdot L_{i-1}\cdot(h_{i-1}+h_i)\cdot\frac{h_i\ \rho_i}{h_{i-1}\ \rho_{i-1}+h_i\ \rho_i}\cdot P_{\rho_i}(1-\theta_i)$$
$$+h_i\cdot L_i\cdot(h_i+h_{i+1})\cdot\frac{h_i\ \rho_i}{h_i\ \rho_i+h_{i+1}\ \rho_{i+1}}\cdot P_{\rho_i}(\theta_i). \qquad (1.15)$$

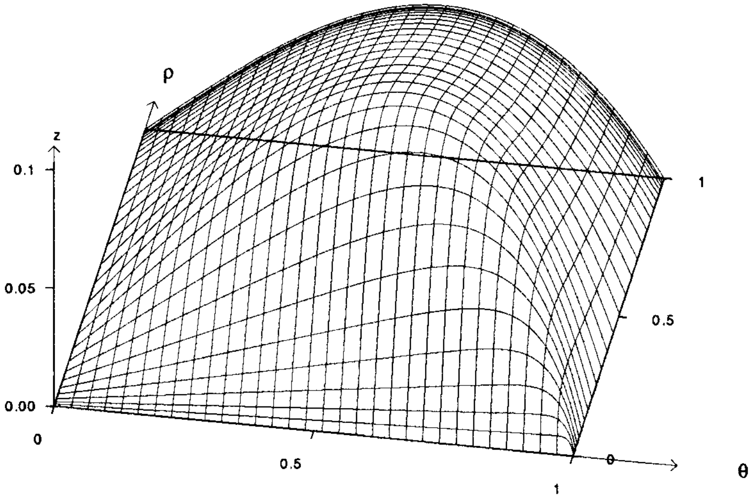

FIG. 1. Graph of $z = -P_\rho(\theta)$.

Equations (1.13), (1.14), and (1.15) are valid for $i = 2, ..., n-1$, but we point out that if we extend the definition of $L_i(\xi)$ to the cases where $i = 0$ and $i = n$ by defining $L_0(\xi) = 0 = L_n(\xi)$, then they remain valid for the extended range $i = 1, ..., n$.

2. OUTLINE OF THE ARGUMENT

Lemmas 4.7–4.9 describe the limiting behaviour of the moments of the spline as $\xi \to \infty$, while Theorem 4.1 describes the limiting behaviour of the slope of the spline as $\xi \to \infty$. As expected, for each x between the knots, $\tau'_\xi(x)$ tends to the slope of the interpolating polygon. Because the latter is ordinarily discontinuous we cannot expect this convergence to be uniform on (x_{i-1}, x_i); however, Theorem 4.1.2 shows the convergence is uniform on $[x_{i-1} + \delta, x_i - \delta]$ for all $\delta > 0$. Theorem 4.1.1 shows that at the interior knots, τ'_ξ tends to a weighted average of the slopes of the interpolating polygon on either side of the knot, with the respective weights being proportional to the tensions on those intervals.

There still remains the possibility that as tensions are suitably increased the derivatives of the resulting family of exponential splines might behave like partial sums S_n of the Fourier series of a periodic step function f, converging pointwise between the discontinuities to the function but exhibiting *Gibbs phenomenon*, in which, although $S_n - f \to 0$ pointwise, $\|S_n - f\|_\infty$ is bounded away from zero. If an analogous behaviour occurred with exponential splines, "spikes" in τ'_ξ would develop near the knots, which could not be "pulled out" by increasing the tension. Theorem 4.3 shows that as long as all the $p_i \to \infty$ this Gibbs-like behaviour cannot occur, in the sense that τ'_ξ remains within an ε-neighbourhood of $\mathrm{Int}(m_i, m_{i+1})$ on all of $[(x_{i-1}+x_i)/2, (x_i+x_{i+1})/2]$.

The principal novelties of the method are:

1. Unlike earlier approaches to exponential splines where the spline is computed from (**p**, Δ and) values of its first or perhaps second derivatives at the knots, these derivatives being found by solving a tridiagonal linear system; here the spline is computed from (**p**, Δ and) a new quantity L_i, see (1.11), with the desirable property that, assuming (1.1a)–(1.1c), its limit as $\xi \to \infty$ and as $\xi \to 0$ always exists. In fact $\lim_{\xi \to \infty} L_i(\xi) = d_i$ and $\lim_{\xi \to 0} L_i(\xi) = y''(x_i)$, where $y(x)$ is the natural cubic spline through Δ.

2. The use of Lemmas 4.4 and 4.5, which show that the root $\theta(\xi)$ of the equation

$$\frac{\sinh[\beta(\xi)(1-\theta(\xi))]}{\sinh[\beta(\xi)\,\theta(\xi)]} = \delta(\xi)\,\frac{(\alpha(\xi)+\beta(\xi))\,\gamma(\xi)}{\alpha(\xi)(\beta(\xi)+\gamma(\xi))}$$

exists, is unique, and converges to $\frac{1}{2}$ as $\alpha(\xi), \beta(\xi), \gamma(\xi) \to \infty$ and $\delta(\xi) \to L \in (0, \infty)$. It is these results which allow Pruess's assumption of uniform boundedness to be dropped.

3. CONNECTIONS WITH EARLIER WORK

Lemma 4.7 is a simpler and stronger version of Pruess's Lemma 2.2 and Lemma 2.3. Theorem 4.1.1 is a correction to a similar claim in Späth (Eq. (25)) and Pruess (Eq. (3.14)). Theorem 4.1.2 is similar to Pruess's Theorem 3. Theorem 4.2.1 is essentially Part 2 of Pruess's Theorem 2, but free from the uniform boundedness constraint. Pruess's Theorem 4 shows that monotone data determine monotone splines for high enough tension, but by using a thumbnail proof which appears to rely on the hidden, non-trivial assumption that the family τ'_ξ is equicontinuous at the knots. It is proved here as Corollary 4.1.5 by a different method. Still, this does not rule out some sort of Gibbs-like phenomenon for τ' in a neighbourhood of a knot. Theorem 4.3 does this.

4. MATHEMATICAL DETAILS

In this section, statements of the results and condensed versions of some of their proofs are given. For detailed proofs of all results see the technical report [3].

LEMMA 4.1. *Let* $\rho = \tanh \sigma / \sigma$ *as in* (1.4). *Define* $Q_\rho \colon [0, 1] \to \mathbb{R}$ *by* (1.7). *Then* (*see Fig*. 2):

$$\lim_{\sigma \to \infty} Q_\rho(0) = 0, \tag{4.1a}$$

$$\lim_{\sigma \to \infty} Q_\rho(1) = 1, \tag{4.1b}$$

$$\lim_{\sigma \to 0} Q_\rho(0) = -\frac{1}{6}, \tag{4.1c}$$

$$\lim_{\sigma \to 0} Q_\rho(1) = \frac{1}{3}, \tag{4.1d}$$

$$\frac{Q_\rho(0)}{\rho} \in \left(-1, -\frac{1}{6}\right], \tag{4.1e}$$

$$Q_\rho(1) \in \left[\frac{1}{3}, 1\right), \tag{4.1f}$$

$Q_\rho(\theta)$ *is a bounded function of* ρ *and* θ. (4.1g)

Proof. This lemma and the three which follow it are fairly routine to prove using well known techniques from algebra and calculus. ▮

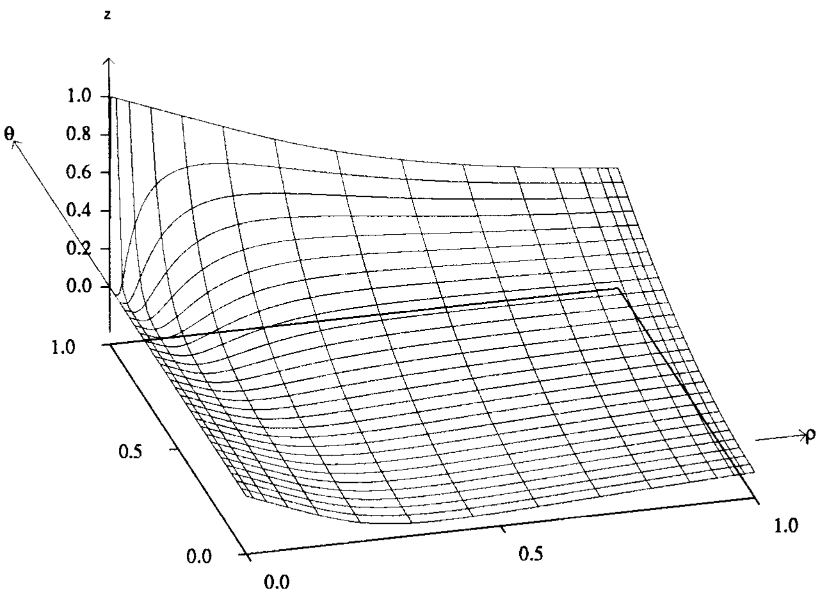

FIG. 2. Graph of $z = Q_\rho(\theta)$.

LEMMA 4.2. *$Q_\rho \to 0$ uniformly on $[0, 1-\varepsilon]$ for every $\varepsilon \in (0, 1)$ and thus $Q_\rho \to 0$ pointwise on $[0, 1)$ as $\rho \to 0$ (see Fig. 2).*

LEMMA 4.3. *For any bounded functions $A(\rho)$ and $B(\rho)$, say of $\rho \in (0, 1)$, define the family of functions $h_\rho : [0, 1] \to \mathbb{R}$ by $h_\rho(\theta) = A(\rho)\, Q_\rho(\theta) + B(\rho)\, Q_\rho(1-\theta)$. Then $h_\rho \to 0$ uniformly on $[\varepsilon, 1-\varepsilon]$ as $\rho \to 0$, and $h_\rho \to 0$ pointwise on $(0, 1)$; however, for all $\rho \in (0, 1)$, except those for which $A(\rho) = 0 = B(\rho)$, h_ρ vanishes for at most one θ.*

LEMMA 4.4. *Let a_n, b_n, c_n, and d_n be sequences of positive reals such that $a_n, b_n, c_n \to \infty$ and $d_n \to L \in (0, \infty)$. Define*

$$g_n = d_n \frac{(a_n + b_n)\, c_n}{a_n(b_n + c_n)}.$$

Then

1. *$g_n \to 0$ implies $1 < 1/g_n < b_n$ for all large n.*
2. *$g_n \to \infty$ implies $1 < g_n < b_n$ for all large n.*

The next lemma shows that for each $\xi > 0$ there is one and only one root $\theta(\xi)$ of the equation

$$\frac{\sinh[\beta(\xi)(1-\theta(\xi))]}{\sinh[\beta(\xi)\, \theta(\xi)]} = \delta(\xi)\, \frac{[\alpha(\xi) + \beta(\xi)]\, \gamma(\xi)}{\alpha(\xi)[\beta(\xi) + \gamma(\xi)]}, \tag{4.2}$$

and it tends to $\frac{1}{2}$ as $\xi \to \infty$.

LEMMA 4.5. *Let α, β, γ, δ be positive valued functions of a variable ξ such that $\lim_{\xi\to\infty} \alpha(\xi) = \infty$, $\lim_{\xi\to\infty} \beta(\xi) = \infty$, $\lim_{\xi\to\infty} \gamma(\xi) = \infty$, and $\lim_{\xi\to\infty} \delta(\xi) = L$ for some $L \in (0, \infty)$. Define $G: (0, \infty) \to (0, \infty)$ by*

$$G(\xi) = \delta(\xi)\, \frac{(\alpha(\xi) + \beta(\xi))\, \gamma(\xi)}{\alpha(\xi)(\beta(\xi) + \gamma(\xi))}.$$

Let $\theta: (0, \infty) \to (0, 1)$ denote the function $\xi \mapsto F_{\tanh \beta(\xi)/\beta(\xi)}(G(\xi))$ (see (1.5) and (1.6)). Then

$$\lim_{\sigma\to\infty} \theta(\xi) = \tfrac{1}{2}.$$

Proof. The existence and uniqueness of a solution $\theta(\xi)$ to (4.2) for each $\xi > 0$ follows from the fact that for each $\rho \in (0, 1]$ it can be shown that f_ρ is a bijection from $(0, 1]$ onto $[0, \infty)$.

Assume if possible that $\theta(\xi) \nrightarrow \frac{1}{2}$; then there would exist a $\vartheta \in [0, 1] - \{\frac{1}{2}\}$ and an increasing unbounded sequence ξ_n such that $\theta(\xi_n) \to \vartheta$. Now $G(\xi_n) \in (0, \infty)$ for all n, so there must exist a subsequence ξ_{n_i} of ξ_n such that one of the following three cases holds: $G(\xi_{n_i}) \to 0$ or ∞ or L' for some $L' \in (0, \infty)$. For brevity let $G_i = G(\xi_{n_i})$, $\theta_i = \theta(\xi_{n_i})$, $\alpha_i = \alpha(\xi_{n_i})$, and so on.

So far we have deduced the following:

(i) $\alpha_i, \beta_i, \gamma_i \to \infty$, $\delta_i \to L \in (0, 1)$ and $G_i = \delta_i \gamma_i (\alpha_i + \beta_i)/(\alpha_i(\beta_i + \gamma_i))$,

(ii) $f_{\tanh \beta_i/\beta_i}(\theta_i) = G_i$,

(iii) $\theta_i \to \vartheta \in [0, 1]$ but $\vartheta \neq \frac{1}{2}$,

(iv) One of the following cases holds:

1. $G_i \to 0$,
2. $G_i \to \infty$,
3. $G_i \to L' \in (0, \infty)$.

We will now show that in each of the three cases in (iv) a contradiction ensues.

First we point out that by (1.6)

$$\theta_i = \frac{1}{2\beta_i} \ln \left(\frac{G_i + e^{\beta_i}}{G_i + e^{-\beta_i}} \right). \tag{4.3}$$

Case 1. Assume $G_i \to 0$, then by Lemma 4.4.1 for i sufficiently large $1 < 1/G_i < \beta_i$, so that

$$\frac{\ln(1 + e^{-\beta_i})}{\beta_i} < \frac{\ln\left(\dfrac{1}{G_i} + e^{-\beta_i}\right)}{\beta_i} < \frac{\ln(\beta_i + e^{-\beta_i})}{\beta_i},$$

and in this inequality the leftmost and rightmost terms tend to 0. Hence

$$\frac{\ln\left(\dfrac{1}{G_i} + e^{-\beta_i}\right)}{\beta_i} \to 0. \tag{4.4}$$

Rearranging (4.3) we have

$$\theta_i = \frac{1}{2}\left[1 - \frac{\ln(1 + (1/(G_i e^{\beta_i})))}{\beta_i} + \frac{\ln((1/G_i) + (1/e^{\beta_i}))}{\beta_i}\right]. \tag{4.5}$$

By (4.4) the last term in (4.5) tends to 0. By Lemma 4.4.1 $1/G_i \in \mathcal{O}(\beta_i)$ so[2] the second term in (4.5) tends to zero also. Therefore we have that $\theta_i \to \frac{1}{2}$ which contradicts $\theta_i \to \vartheta \neq \frac{1}{2}$.

Case 2. Assume $G_i \to \infty$. Now the proof proceeds similarly to Case 1, but using Lemma 4.4.2.

Case 3. Assume $G_i \to L'$ for some $L' > 0$. Rewriting (4.3) gives

$$\theta_i = \frac{1}{2}\left[1 + \frac{\ln(1+(G_i/e^{\beta_i}))}{\beta_i} - \frac{\ln(1+(1/(G_i e^{\beta_i})))}{\beta_i} - \frac{\ln G_i}{\beta_i}\right].$$

Clearly all terms but the first in the square brackets tend to 0 again, resulting in a contradiction. ▮

LEMMA 4.6. *Let $\xi \mapsto A_\xi$ be an $n \times n$ matrix valued function of a real variable ξ. Let $a \in [-\infty, \infty]$. Suppose A is nonsingular and $\lim_{\xi \to a} A_\xi = A$ component-wise. Then A_ξ is nonsingular for ξ near a, and for any fixed $\mathbf{c} \in \mathbb{R}^n$*

$$\lim_{\xi \to a} A_\xi^{-1}\mathbf{c} = A^{-1}\mathbf{c}.$$

Proof. This result follows from some matrix inequality manipulations. ▮

LEMMA 4.7. *Consider the exponential spline defined by* (1.2)–(1.3h). *Assume* (1.1a)–(1.1c) *hold. Let* $\mathbf{L}$ *be the solution to* (1.12); *then, for* $i = 1, ..., n-1$,

$$\lim_{\xi \to \infty} L_i(\xi) = d_i.$$

In addition, if $y(x)$ is the natural cubic spline through the same knot sequence, then

$$\lim_{\xi \to 0} L_i(\xi) = y''(x_i).$$

Proof. The proof to this result uses Lemma 4.1 and Lemma 4.6. ▮

LEMMA 4.8. *If $\mathbf{d} \neq \mathbf{0}$ then $\|\mathbf{L}(\xi)\|$ is bounded away from* 0 *for all $\xi \geqslant 0$.*

Proof. The proof to this result requires some inequality manipulations on (1.12) and the use of (4.1e) and (4.1f). ▮

LEMMA 4.9. *Consider the exponential spline defined by* (1.2)–(1.3h) *and satisfying* (1.1a)–(1.1c). *For all* $i = 1, ..., n-1$,

$$M_i(\xi) \text{ is bounded if and only if } d_i = 0.$$

[2] In this paper the notations $\mathcal{O}$, Ω, and Θ for order of growth are those of Knuth [5] as enhanced by Brassard [2].

Proof. The proof starts with (1.11) and uses (1.12), (4.1f), and Lemma 4.7. ∎

THEOREM 4.1. *Let $\tau_\xi(x)$ be the family of exponential splines interpolating the knot sequence Δ, defined by* (1.2)–(1.3h) *and subject to a tension strategy* **p** *satisfying* (1.1a)–(1.1c). *Then the following hold*:

1. $$\lim_{\xi \to \infty} \left(\tau'_\xi(x_i) - \left[\frac{p_i(\xi)}{p_i(\xi)+p_{i+1}(\xi)} m_i + \frac{p_{i+1}(\xi)}{p_i(\xi)+p_{i+1}(\xi)} m_{i+1} \right] \right) = 0,$$

$i = 1, ..., n-1.$

Note then that even in the event $\lim_{\xi \to \infty} (p_i(\xi)/(p_i(\xi)+p_{i+1}(\xi)))$ *fails to exist, we can at least say that, for any* $\varepsilon > 0$, $\tau'_\xi(x_i) \in \mathrm{Int}(m_i - \varepsilon,\ m_{i+1} - \varepsilon,\ m_i + \varepsilon,\ m_{i+1} + \varepsilon)$ *for all sufficiently large* ξ.

2. *For all small* $\delta > 0$, $\tau'_\xi \to m_i$ *uniformly on* $[x_{i-1} + \delta,\ x_i - \delta]$ *as* $\xi \to \infty$ *for each* $i = 1, ..., n$.

3. *For all small* $\delta > 0$, $\tau'_\xi \to m_1$ *uniformly on* $[x_0, x_1 - \delta]$ *and* $\tau'_\xi \to m_n$ *uniformly on* $[x_{n-1} + \delta, x_n]$ *as* $\xi \to \infty$.

4. *If* $m_i = m_{i+1}$ *for some* $i = 1, ..., n-1$, *then for all small* $\delta > 0$, $\tau'_\xi \to m_i$ *uniformly on* $[x_{i-1} + \delta,\ x_{i+1} - \delta]$ *as* $\xi \to \infty$.

5. $\lim_{\xi \to \infty} \tau'_\xi(x) = m_i$ *for each* $i = 1, ..., n$ *and each* $x \in (x_{i-1}, x_i)$.

Proof. Part 1: Assume $x = x_i$ for some $i = 1, ..., n-1$, then $\cosh(\sigma_i(1-\theta_i))/\sinh \sigma_i = 1/\sinh \sigma_i \to 0$ as $\xi \to \infty$. Similarly $\cosh(\sigma_i \theta_i)/\sinh \sigma_i = \coth \sigma_i \to 1$ as $\xi \to \infty$. Therefore, by (1.13), remembering that $d_i = (m_{i+1} - m_i)/(h_i + h_{i+1})$, we obtain, with a bit of rearranging,

$$\begin{aligned}
&\tau'_\xi(x_i) - \left[\frac{p_i}{p_i + p_{i+1}} m_i + \frac{p_{i+1}}{p_i + p_{i+1}} m_{i+1} \right] \\
&\quad = -L_{i-1} \cdot (h_{i-1} + h_i) \cdot \frac{h_i\ \rho_i}{h_i\ \rho_i + h_{i-1}\ \rho_{i-1}} \cdot \frac{Q_{\rho_i}(0)}{\rho_i} \cdot \rho_i \\
&\qquad + L_i \cdot (h_i + h_{i+1}) \cdot \frac{h_i\ \rho_i}{h_i\ \rho_i + h_{i+1}\ \rho_{i+1}} \cdot (Q_{\rho_i}(1) - 1) \\
&\qquad + (h_i + h_{i+1}) \cdot \frac{h_i\ \rho_i}{h_i\ \rho_i + h_{i+1}\ \rho_{i+1}} \cdot (L_i - d_i) \\
&\qquad + (h_i + h_{i+1}) \cdot \frac{h_i\ \rho_i}{h_i\ \rho_i + h_{i+1}\ \rho_{i+1}} \cdot \frac{d_i}{\tanh \sigma_i} \cdot \frac{p_i}{p_i + p_{i+1}} \\
&\qquad \cdot (\tanh \sigma_i - \tanh \sigma_{i+1}).
\end{aligned}$$

In every term on the right-hand side of this equation the last factor tends to 0 as $\xi \to \infty$ while the other factors remain bounded. Note that this result does not imply that $\lim_{\xi \to \infty} \tau'_\xi(x_i)$ exists.

Part 2: This follows by applying Lemmas 4.7 and 4.3 to Eq. (1.13).

Part 3: This follows by applying Lemmas 4.7 and 4.2 to Eq. (1.13).

Part 4: Assume $m_i = m_{i+1}$, then $d_i = 0$. Now apply Lemma 4.2, Lemma 4.7, and (4.1g) to Eq. (1.13).

Part 5: Proof of this part is an immediate consequence of Parts 2 and 3 above. ∎

THEOREM 4.2. *Let* $\tau_\xi(x)$ *be the exponential spline defined by* (1.2)–(1.3h) *and subject to a tension strategy* **p** *satisfying* (1.1a)–(1.1c); *then, for* $i = 2, ..., n-1$:

1. *If* $d_i d_{i-1} > 0$ (*that is,* m_{i-1}, m_i, m_{i+1} *form a strictly monotone sequence*), *then for all sufficiently large* ξ, $\tau'_\xi(x)$ *attains its maximum and minimum values on* $[x_{i-1}, x_i]$ *at the endpoints of this interval and* $\tau''_\xi(x)$ *does not change sign or vanish on* $[x_{i-1}, x_i]$.

2. *If* $d_i d_{i-1} < 0$ (*that is* m_{i-1}, m_i, m_{i+1} *form a strictly monotone sequence*) *then for all sufficiently large* ξ:

(a) $\tau''_\xi(x)$ *vanishes at exactly one point* $x = \tilde{x}$ *on* $[x_{i-1}, x_i]$, *and in addition,* $\tilde{x}$ *lies in the interior of* $[x_{i-1}, x_i]$;

(b) τ''_{ξ_i} *changes sign at* $x = \tilde{x}$, *so that* τ'_{ξ_i} *attains a local extremum there, but at no other point in* (x_{i-1}, x_i);

(c) $\lim_{\xi \to \infty} (\tilde{x}, \tau_\xi(\tilde{x})) = ((x_{i-1} + x_i)/2, (y_{i-1} + y_i)/2)$.

Proof. Part 1: By hypothesis d_i and d_{i-1} have the same sign; therefore, by Lemma 4.7 for all sufficiently large ξ, $L_i(\xi)$ and $L_{i-1}(\xi)$ have the same (fixed, non-zero) sign. Therefore, from (1.14) we get for all such large ξ that $\tau''_\xi(x)$ has the same sign as L_i and L_{i-1} on all of $[x_{i-1}, x_i]$, and hence $\tau'_\xi(x)$ has no local extrema on the interior of $[x_{i-1}, x_i]$.

Part 2: By Lemma 4.7 and hypothesis, for sufficiently high ξ (say $\xi > N$) we can be sure that L_i and L_{i-1} have the same (non-zero) signs as d_i and d_{i-1} respectively, and so

$$-\frac{L_i(\xi)}{L_{i-1}(\xi)} > 0. \tag{4.6}$$

Let $x \in (x_{i-1}, x_i)$ so that $\theta_i \in (0, 1)$ and (1.14) can be rewritten

$$\tau''_\xi(x) = L_{i-1}(\xi) \cdot \frac{h_{i-1} + h_i}{h_{i-1}\, \rho_{i-1} + h_i\, \rho_i} \cdot \frac{\sinh(\theta_i \sigma_i)}{\sinh \sigma_i}$$

$$\cdot \left[\frac{L_i(\xi)(h_i + h_{i+1})}{L_{i-1}(\xi)(h_{i-1} + h_i)} \cdot \frac{\left(\dfrac{h_i \tanh \sigma_i}{h_{i-1}\, \rho_{i-1}} + \sigma_i\right) \dfrac{h_i \tanh \sigma_i}{h_{i+1}\, \rho_{i+1}}}{\dfrac{h_i \tanh \sigma_i}{h_{i-1}\, \rho_{i-1}} \left(\sigma_i + \dfrac{h_i \tanh \sigma_i}{h_{i+1}\, \rho_{i+1}}\right)} + f_{\rho_i}(\theta_i) \right] \tag{4.7}$$

which for all $\xi > N$ changes sign in (x_{i-1}, x_i) exactly where

$$f_{\rho_i}(\theta_i) - \left[-\frac{L_i}{L_{i-1}} \cdot \frac{h_i + h_{i+1}}{h_{i-1} + h_i} \cdot \frac{\left(\dfrac{h_i \tanh \sigma_i}{h_{i-1}\, \rho_{i-1}} + \sigma_i\right) \dfrac{h_i \tanh \sigma_i}{h_{i+1}\, \rho_{i+1}}}{\dfrac{h_i \tanh \sigma_i}{h_{i-1}\, \rho_{i-1}} \left(\sigma_i + \dfrac{h_i \tanh \sigma_i}{h_{i+1}\, \rho_{i+1}}\right)} \right] \tag{4.8}$$

changes sign. For each $\xi > N$, by (4.6), the bracketed term in (4.8) is positive, so we know by the fact that $f_\rho\colon (0, 1] \to [0, \infty)$ is onto and strictly monotone that this sign change occurs for one and only one $\theta_i = \tilde{\theta}_i \in (0, 1)$. As $\xi \to \infty$ clearly $(h_i \tanh \sigma_i)/(h_{i-1}\, \rho_{i-1})$, $(h_i \tanh \sigma_i)/(h_{i+1}\, \rho_{i+1})$, and $\sigma_i \to \infty$, so by Lemma 4.5 we have $\tilde{\theta}_i \to \frac{1}{2}$ and hence $\tilde{x} \to (x_{i-1} + x_i)/2$. ∎

THEOREM 4.3. *Let $\tau_\xi(x)$ be the family of exponential splines through knot sequence Δ, defined by* (1.2)–(1.3h) *and a tension strategy* **p** *which satisfies* (1.1a)–(1.1c), *then for each $\varepsilon > 0$, for all sufficiently large ξ:*

(i) $\tau'_\xi \in \mathrm{Int}(m_1 \pm \varepsilon)$ *on* $[x_0, (x_0 + x_1)/2]$,

(ii) $\tau'_\xi \in \mathrm{Int}(m_i \pm \varepsilon,\ m_{i+1} \pm \varepsilon)$ *on* $[(x_{i-1} + x_i)/2,\ (x_i + x_{i+1})/2]$ *if* $1 \leqslant i \leqslant n-1$,

(iii) $\tau'_\xi \in \mathrm{Int}(m_n \pm \varepsilon)$ *on* $[(x_{n-1} + x_n)/2,\ x_n]$.

Proof. We divide the proof into cases and show that for all sufficiently large ξ:

1. If $x \in [x_0, (x_0 + x_1)/2]$ then $\tau'_\xi(x) \in (m_1 - \varepsilon, m_1 + \varepsilon)$.

2. If $x \in [x_i, (x_i + x_{i+1})/2]$ and $d_i = 0$ and $i = 1, ..., n-1$ then we have $\tau'_\xi(x) \in (m_i - \varepsilon, m_i + \varepsilon)$.

3. If $x \in [x_i, (x_i + x_{i+1})/2]$ and $d_i \cdot d_{i+1} < 0$ and $i = 1, ..., n-2$ then $\tau'_\xi(x) \in \mathrm{Int}(m_i \pm \varepsilon, m_{i+1} \pm \varepsilon)$.

4. If $x \in [x_i, (x_i + x_{i+1})/2]$ and $d_i \cdot d_{i+1} > 0$ and $i = 1, ..., n-2$ then $\tau'_\xi(x) \in \mathrm{Int}(m_i \pm \varepsilon, m_{i+1} \pm \varepsilon)$.

5. If $x \in [x_i, (x_i + x_{i+1})/2]$, and $d_i \neq 0$ and $d_{i+1} = 0$ and $i = 1, ..., n-2$ then $\tau'_\xi(x) \in \mathrm{Int}(m_i \pm \varepsilon, m_{i+1} \pm \varepsilon)$.

6. If $x \in [x_{n-1}, (x_{n-1}+x_n)/2]$ and $d_{n-1} \neq 0$ then we have that $\tau'_\xi(x) \in \mathrm{Int}(m_{n-1} \pm \varepsilon, m_n \pm \varepsilon)$.

Here we have appealed to symmetry and omitted the cases that correspond to the right half of intervals.

Case 1. Follows immediately from Theorem 4.1.3.

Case 2. In this case $m_i = m_{i+1}$; therefore, by Theorem 4.1.4, $\tau'_\xi \to m_i$ uniformly on $[x_i, (x_i+x_{i+1})/2]$ and the result follows.

Case 3. By Theorem 4.2.2 we know that there exists a $W > 0$ such that for all $\xi > W$ it is the case that τ'_ξ attains a local extremum for exactly one point $\tilde{x}$ within (x_i, x_{i+1}), and if W is chosen high enough we can be sure $\tilde{x}$ lies in the central half of the interval $[x_i, x_{i+1}]$. By Theorem 4.1.2, given any $\varepsilon > 0$, we can say that there exists a $U > 0$ such that for all $\xi > U$ we have $\tau'_\xi \in (m_{i+1} - \varepsilon, m_{i+1} + \varepsilon)$ on the central half of $[x_i, x_{i+1}]$. Therefore, for all $\xi > \max(W, U)$ the extreme values of τ'_ξ on $[x_i, (x_i + x_{i+1})/2]$ lie in the set $\{\tau'_\xi(x_i)\} \cup (m_{i+1} - \varepsilon, m_{i+1} + \varepsilon)$. By Theorem 4.1.1 there exists a $Z > 0$ such that for all $\xi > Z$, $\tau'_\xi(x_i) \in \mathrm{Int}(m_i \pm \varepsilon, m_{i+1} \pm \varepsilon)$. Consequently, on $[x_i, (x_i + x_{i+1})/2]$, for $\xi > \max(W, U, Z)$ we have that $\tau'_\xi(x) \in \mathrm{Int}(m_i \pm \varepsilon, m_{i+1} \pm \varepsilon)$.

Case 4. For all sufficiently large ξ the following three statements hold. By Theorem 4.1.1, $\tau'_\xi(x_i) \in \mathrm{Int}(m_i \pm \varepsilon, m_{i+1} \pm \varepsilon)$. From Theorem 4.1.5, $\tau'_\xi((x_i + x_{i+1})/2) \in \mathrm{Int}(m_{i+1} \pm \varepsilon)$. From Theorem 4.2.1 the extreme values of τ'_ξ on $[x_i, (x_i + x_{i+1})/2]$ are at its endpoints. Consequently, on $[x_i, (x_i + x_{i+1})/2]$, for all ξ sufficiently large $\tau'_\xi(x) \in \mathrm{Int}(m_i \pm \varepsilon, m_{i+1} \pm \varepsilon)$.

Case 5. For some sufficiently large $T > 0$, for all $\xi > T$ by Lemma 4.7 L_i maintains the same (non-zero) sign as d_i. By Eq. (4.7) and the fact that $f_\rho \colon (0, 1] \to [0, \infty)$ is a bijection, it follows that for each such ξ there is at most one $x = \tilde{x}_\xi \in [x_i, (x_i + x_{i+1})/2]$ for which τ''_ξ vanishes (none if L_i and L_{i+1} have the same sign). For those ξ for which such an $\tilde{x}_\xi$ exists, define $\theta_\xi = (\tilde{x}_\xi - x_i)/h_{i+1}$. Then $\xi \mapsto \theta_\xi$ is a partial function from (T, ∞) to $[0, \frac{1}{2}]$.

First we show that there exists a $\delta \in (0, (x_{i+1} - x_i)/2)$ and a $V > T$ such that for all $\xi > V$ we have $\tau''_\xi \neq 0$ on $[x_i, x_i + \delta]$. Suppose not, then there would exist an infinite sequence $\eta_j > V$ such that $\eta_j \to \infty$ and $\tau''_{\eta_j}(\tilde{x}_{\eta_j}) = 0$ and $\tilde{x}_{\eta_j} \to x_i$ (i.e., $\theta_{\eta_j} \to 0$). Therefore, by (1.14):

$$\begin{aligned} 0 &= \tau''_{\eta_j}(\tilde{x}_{\eta_j}) \\ &= L_i(\eta_j) \cdot \frac{h_i + h_{i+1}}{h_i\ \rho_i(\eta_j) + h_{i+1}\ \rho_{i+1}(\eta_j)} \cdot \frac{\sinh(\sigma_{i+1}(\eta_j)(1 - \theta_{\eta_j}))}{\sinh \sigma_{i+1}(\eta_j)} \\ &\quad + L_{i+1}(\eta_j) \cdot \frac{h_{i+1} + h_{i+2}}{h_{i+1}\ \rho_{i+1}(\eta_j) + h_{i+2}\ \rho_{i+2}(\eta_j)} \cdot \frac{\sinh(\sigma_{i+1}(\eta_j)\theta_{\eta_j})}{\sinh \sigma_{i+1}(\eta_j)}. \end{aligned} \tag{4.9}$$

By (4.9), since $L_i(\eta_j) \to d_i \neq 0$, there exists a $R > T$ such that for all $\eta_j > R$ we have that $\theta_{\eta_j} \neq 0$, and thus $(\sinh(\sigma_{i+1}(\eta_j)(1-\theta_{\eta_j})))/(\sinh(\sigma_{i+1}(\eta_j)\,\theta_{\eta_j}))$ is defined for all sufficiently large j. Since M_{i+1} is bounded by Lemma 4.9, we have by (4.9) and (1.11):

$$\lim_{j\to\infty} f_\rho(\theta_{\eta_j}) = \lim_{j\to\infty} \frac{\sinh(\sigma_{i+1}(\eta_j)(1-\theta_{\eta_j}))}{\sinh(\sigma_{i+1}(\eta_j)\,\theta_{\eta_j})}$$
$$= -\lim_{j\to\infty} \frac{M_{i+1}(\eta_j)}{L_i(\eta_j)} \cdot \left[\frac{h_i}{h_i+h_{i+1}}\,\rho_i(\eta_j) + \frac{h_{i+1}}{h_i+h_{i+1}}\,\rho_{i+1}(\eta_j)\right] = 0. \tag{4.10}$$

It is not hard to show that we have $f_\rho(\theta) \to \infty$ as $\theta \to 0$ and $\sigma \to \infty$, contradicting (4.10).

At this point we know that for all sufficiently large ξ the extreme values of τ'_ξ on $[x_i, (x_i + x_{i+1})/2]$ are attained only for $x \in \{x_i\} \cup [x_i + \delta,$

EXPONENTIAL SPLINES

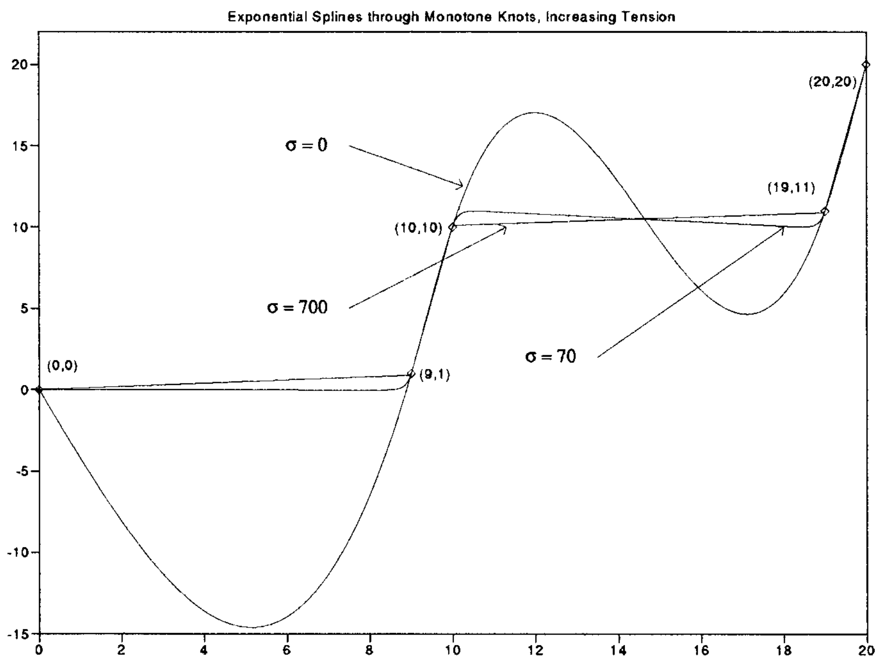

FIGURE 3

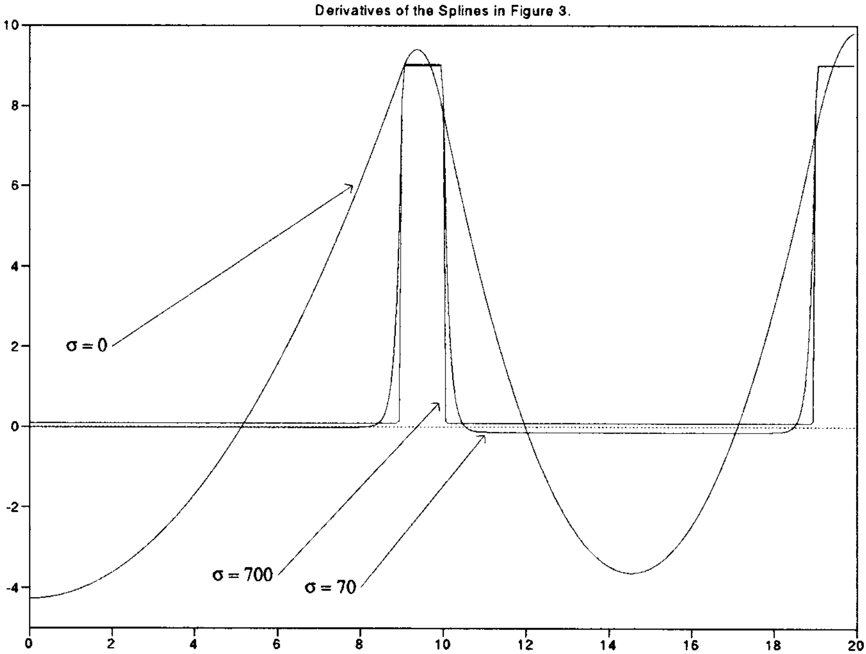

FIGURE 4

$(x_i + x_{i+1})/2]$. However, $\tau'_\xi \to m_{i+1}$ uniformly on $[x_i + \delta, (x_i + x_{i+1})/2]$ by Theorem 4.1.4, while Theorem 4.1.1 tells us $\tau'_\xi(x_i) \in \text{Int}(m_i \pm \varepsilon, m_{i+1} \pm \varepsilon)$. Thus we conclude, in this case, that for each $\varepsilon > 0$ there exists an $H > 0$ such that $\tau'_\xi(x) \in \text{Int}(m_i \pm \varepsilon, m_{i+1} \pm \varepsilon)$ when $\xi > H$.

Case 6. The proof here is virtually identical with the previous case owing to the fact that, as was remarked when it was derived, Eq. (1.14) remains valid when $i = n$ if we define $L_n(\xi) = 0$ and $d_n = 0$. ▮

COROLLARY 4.1. *Under the hypotheses of the theorem, if* $m_i > 0$ *for all* $i = 1, 2, ..., n$ *then* $\tau_\xi(x)$ *is strictly increasing for all sufficiently large* ξ.

5. AN EXAMPLE

See Fig. 3. This example shows the effect of increasing σ_i on the exponential spline (with uniform tension) through a monotone but serpentine knot sequence. Note that zero tension results in the natural cubic spline, but that for sufficiently high tension the spline becomes monotone. Note in Fig. 4, the absence of any Gibbs-like phenomenon in $\tau'(x)$ as σ increases.

6. CONJECTURES AND QUESTIONS

CONJECTURES. 1. *If* $d_i \neq 0$ *for some i then* $|L_i(\xi)|$ *is bounded away from* 0 *for all* ξ, (*a slightly stronger result than Lemma* 4.8).

2. $\mathbf{R}_\xi$ *is never singular for any choice of knots and tensions. Hence*:

(a) *for each fixed* Δ *and* $\mathbf{p}$ *there exists an* $M > 0$ *such that* $\|\mathbf{L}\| < M$ *for all* $\xi \geqslant 0$.

(b) *Equation* (1.12) *has one and only one solution for all* Δ, $\mathbf{p}$, *and* ξ.

3. $\mathbf{R}_\xi$ *is well-conditioned for any choice of* Δ, $\mathbf{p}$, *and* ξ.

QUESTIONS. 1. *What happens when only some of the tensions tend to infinity, or some other weakening of* (1.1a)–(1.1c) *is allowed*?

2. *What is an explicit expression for* $\mathbf{p}$ *in terms of* Δ *which, without overkill, is sufficient to make the spline through monotone knots monotone*?

3. *The results in this paper concern* τ'. *What results about* τ'' *can be derived by similar methods*?

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